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Correlation and Information in Quantum Channels

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Quantum and classical correlations in quantum channels are investigated by means of an entangled pure state and a separable state which is closest to an entangled pure state. The coherent information and the separable information are used to estimate how much correlation is transmitted through a quantum channel. As the examples, the linear dissipative channel of qubits and the quantum erasure channel are considered.

KEY WORDS: entanglement; classical correlation; information; quantum channel.

1. INTRODUCTION

In quantum information processing (Nielsen and Chuang, 2000), change of quantum states is described by a trace-preserving completely positive map which is called a quantum channel or a quantum operation. Hence it is important to investigate how much quantum and classical correlation in quantum states change under the influence of quantum channels. For a general quantum state, it is very difficult to separate between quantum correlation and classical correlation (Henderson and Vedral, 2001). Hence a pure entangled state $|\Psi^{AB}\rangle$ is used for investigating quantum correlation and a separable state $\hat{\rho}_{\mathrm{s}}^{AB}$ closest to a pure entangled state $|\Psi^{AB}\rangle$ is considered to investigate classical correlation. In this paper, using information-theoretical quantitites, we investigate how much correlation in quantum states $|\Psi^{AB}\rangle$ and $\hat{\rho}_{\mathrm{s}}^{AB}$ changes under the influence of quantum channels. In Section 2, we briefly summarize a quantum channel, an entangled pure state and a separable state closest to an entangled pure state. In Sections 3 and 4, the basic properties of the coherent information and the separable information are reviewed. In Section 5, the classical correspondence of the coherent information and the separable information is given. In Section 6, the degradation of correlations cased by quantum channel is investigated. In Section 7, we consider correlations between an input

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and output systems of a quantum channel. In Sections 8 and 9, we investigate the linear dissipative channel of qubits and the quantum erasure channels and we calculate the coherent information and the separable information. In Section 10, we give the concluding remarks.

2. ENTANGLED STATE AND SEPARABLE STATE

We consider a quantum state described by a density operator $\hat{\rho}^A$, the support space of which is assumed to be an N-dimensional Hilbert space $\mathcal{H}^A$. Introducing an auxiliary Hilbert space $\mathcal{H}^B$, we can extend a quantum state $\hat{\rho}^A$ to a quantum state $\hat{\rho}^{AB}$ defined on a Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$, the partial trace of which yields the original quantum state $\hat{\rho}^A$, that is, $\mathrm{Tr}_B \hat{\rho}^{AB} = \hat{\rho}^A$. Of course, such an extension is not uniquely determined. We denote the spectral decomposition of the quantum state $\hat{\rho}^A$ as $\hat{\rho}^A = \sum_{k=1}^N \lambda_k |\psi_k^A\rangle\langle\psi_k^A| (\lambda_k > 0$ for all k), where $\langle\psi_j^A|\psi_k^A\rangle = \delta_{jk}$ and $\sum_{k=1}^N |\psi_k^A\rangle\langle\psi_k^A| = \hat{1}^A$ with $\hat{1}^A$ being an identity operator defined on the Hilbert space $\mathcal{H}^A$. One of the extended quantum states $\hat{\rho}^{AB}$ is a purification of the quantum state $\hat{\rho}^A$ (Schumacher, 1996), which is given by

$$\hat{\rho}_{\mathrm{e}}^{AB} = |\Psi^{AB}\rangle\langle\Psi^{AB}|, \tag{1}$$

with the entangled state $\Psi^{AB}\rangle$,

$$|\Psi^{AB}\rangle = \sum_{k=1}^{N} \sqrt{\lambda_k} |\psi_k^A\rangle \otimes |\psi_k^B\rangle, \tag{2}$$

where $\{|\psi_k^B\rangle | k = 1, 2, \ldots, N\}$ is an orthonormal system of the Hilbert space $\mathcal{H}^B$. It is obvious that $\mathrm{Tr}_B |\Psi^{AB}\rangle\langle\Psi^{AB}| = \hat{\rho}^A$ is satisfied.

When a distance between two quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ is measured by means of the quantum relative entropy (Ohya and Petz, 1993),

$$S(\hat{\rho}_1|\hat{\rho}_2) = \mathrm{Tr}[\hat{\rho}_1(\log\hat{\rho}_1 - \log\hat{\rho}_2)], \tag{3}$$

the separable quantum state $\hat{\rho}_{\mathrm{s}}^{AB}$ closest to the entangled pure state $\hat{\rho}_{\mathrm{e}}^{AB} = |\Psi^{AB}\rangle$ $\langle\Psi^{AB}|$ is given by

$$\hat{\rho}_{\mathrm{s}}^{AB} = \sum_{k=1}^{N} \lambda_k |\psi_k^A\rangle\langle\psi_k^A| \otimes |\psi_k^B\rangle\langle\psi_k^B|, \tag{4}$$

which satisfies the equality $S(\hat{\rho}_{\mathrm{e}}^{AB}|\hat{\rho}_{\mathrm{s}}^{AB}) = \min_{\hat{\sigma}^{AB} \in \mathcal{S}_{AB}} S(\hat{\rho}_{\mathrm{e}}^{AB}|\hat{\sigma}^{AB})$ (Vedral and Plenio, 1988), where $\mathcal{S}_{AB}$ is the set of all separable quantum states defined on the Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$. Since the quantum state $\hat{\rho}_{\mathrm{s}}^{AB}$ satisfies the relation $\mathrm{Tr}_B \hat{\rho}_{\mathrm{s}}^{AB} = \hat{\rho}^A$, the separable state $\hat{\rho}_{\mathrm{s}}^{AB}$ is also the extension of the quantum state $\hat{\rho}^A$. The systems A and B in the quantum state $\hat{\rho}_{\mathrm{e}}^{AB}$ are entangled, i.e., quantum mechanically correlated while they are classically correlated in the quantum state $\hat{\rho}_{\mathrm{s}}^{AB}$.

A quantum channel (or a quantum operation) that transforms a quantum state into another is mathematically represented by a trace-preserving completely positive map $\hat{\mathscr{L}}$. When the system A is transmitted through a quantum channel $\hat{\mathscr{L}}^A$ and the system B remains unchanged, the extended quantum states $\hat{\rho}_{\mathrm{e}}^{AB}$ and $\hat{\rho}_{\mathrm{s}}^{AB}$ becomes

$$\hat{\rho}_{\mathrm{e\,out}}^{AB} = (\hat{\mathscr{L}}^A \otimes \hat{\mathscr{I}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|, \tag{5}$$

$$\hat{\rho}_{\mathrm{s\,out}}^{AB} = \sum_{k=1}^{N} \lambda_k \hat{\mathscr{L}}^A \big|\psi_k^A\big\rangle\big\langle\psi_k^A\big| \otimes \big|\psi_k^B\big\rangle\big\langle\psi_k^B\big|, \tag{6}$$

where $\hat{\mathscr{I}}^B$ is an identity map for operators of the system B. It is an important task is quantum information processing to estimate how much entanglement and classical correlation between the systems A and B are preserved (or degraded) in the transmission through a quantum channel $\hat{\mathscr{L}}^A$.

A trace-preserving completely positive map $\hat{\mathscr{L}}^A$ is frequently expressed in the two different but equivalent forms (Kraus, 1983; Schumacher, 1996). One is called the Kraus representation (or the operator-sum representation) which is given by

$$\hat{\mathscr{L}}^A \hat{X}^A = \sum_{\mu} \hat{A}_\mu \hat{X}^A \hat{A}_\mu^\dagger, \tag{7}$$

for an arbitrary operator $\hat{X}^A$, where the operator $\hat{A}_\mu$ in the representation satisfies the relation $\sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{1}^A$. The other is called the unitary representation, which is given by

$$\hat{\mathscr{L}}^A \hat{X}^A = \mathrm{Tr}_E[\hat{U}^{A\mathrm{E}}(\hat{X}^A \otimes |0^{\mathrm{E}}\rangle\langle 0^{\mathrm{E}}|)\hat{U}^{AE\dagger}], \tag{8}$$

where $|0^{\mathrm{E}}\rangle$ is a quantum state of an environmental system and $\hat{U}^{A\mathrm{E}}$ is a unitary operator defined on the Hilbert space $\mathscr{H}^A \otimes \mathscr{H}^E$ with $\mathscr{H}^E$ being the Hilbert space of the environment. Using the unitary representation, we can express the quantum state $\hat{\rho}_{\mathrm{e\,out}}^{AB}$ as

$$\hat{\rho}_{\mathrm{e\,out}}^{AB} = \mathrm{Tr}_E \big|\Psi_{\mathrm{out}}^{AB\mathrm{E}}\big\rangle\big\langle\Psi_{\mathrm{out}}^{AB\mathrm{E}}\big|, \tag{9}$$

with

$$\big|\Psi_{\mathrm{out}}^{AB\mathrm{E}}\big\rangle = (\hat{U}^{A\mathrm{E}} \otimes \hat{1}^B)|\Psi^{AB\mathrm{E}}\rangle, \tag{10}$$

and

$$|\Psi^{AB\mathrm{E}}\rangle = |\Psi^{AB}\rangle \otimes |0^{\mathrm{E}}\rangle. \tag{11}$$

The change of the quantum state caused by the quantum channel is evaluated by

$$F_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \langle\Psi^{AB}|[(\hat{\mathscr{L}}^A \otimes \hat{\mathscr{I}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|]|\Psi^{AB}\rangle, \tag{12}$$

$$\bar{F}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \sum_{k=1}^{N} \lambda_k \big\langle\psi_k^A\big|\big(\hat{\mathscr{L}}^A\big|\psi_k^A\big\rangle\big\langle\psi_k^A\big|\big)\big|\psi_j^A\big\rangle. \tag{13}$$

The former is called the entanglement fidelity and the latter is called the average fidelity (Schumacher, 1996). The entanglement fidelity is no greater than the average fidelity, namely, the inequality $F_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) \leq \bar{F}(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ holds. For the entangled pure state $\hat{\rho}_{\mathrm{e}}^{AB} = |\Psi^{AB}\rangle\langle\Psi^{AB}|$, the quantum information-theoretical properties of the transformation $\hat{\rho}_{\mathrm{e}}^{AB} \to (\hat{\mathscr{L}}^A \otimes \hat{\mathscr{I}}^R)\hat{\rho}_{\mathrm{e}}^{AB}$ have been investigated (Schumacher and Nielsen, 1996). The reversibility of such a state transformation is closely related to the quantum error correction (Knill and Laflamme, 1997).

3. COHERENT INFORMATION AND QUANTUM ENTANGLEMENT

We consider the case that the system A in the entangled pure quantum state $\hat{\rho}_{\mathrm{e}} = |\Psi^{AB}\rangle\langle\Psi^{AB}|$ is sent through noisy quantum channel described by a trace-preserving completely positive map $\hat{\mathscr{L}}^A$. The properties of the transmitted state $\hat{\rho}_{\mathrm{e\,out}}^{AB} = (\hat{\mathscr{L}}^A \otimes \hat{\mathscr{I}}^B)\hat{\rho}_{\mathrm{e}}$ have been investigated in detail (Schumacher, 1996; Schumacher and Nielsen, 1996). Hence we briefly summarized the results in this section. The von Neumann entropies of the input and output states of the system A are

$$S(\hat{\rho}^A) = -\mathrm{Tr}[\hat{\rho}^A \log \hat{\rho}^A], \tag{14}$$

$$S\left(\hat{\rho}_{\mathrm{out}}^A\right) = -\mathrm{Tr}\left[\hat{\rho}_{\mathrm{out}}^A \log \hat{\rho}_{\mathrm{out}}^A\right], \tag{15}$$

where $\hat{\rho}_{\mathrm{out}}^A = \mathrm{Tr}_B \hat{\rho}_{\mathrm{e\,out}}^{AB}$. Since $\hat{\rho}_{\mathrm{e}}^{AB}$ is a pure state and the system B remains unchanged, we obtain the relation,

$$S(\hat{\rho}^A) = S(\hat{\rho}^B) = S\left(\hat{\rho}_{\mathrm{out}}^B\right), \tag{16}$$

where $\hat{\rho}^B = \mathrm{Tr}_A \hat{\rho}_{\mathrm{e}}^{AB}$ and $\hat{\rho}_{\mathrm{out}}^B = \mathrm{Tr}_A \hat{\rho}_{\mathrm{e\,out}}^{AB}$. The entropy exchange of the quantum channel $\hat{\mathscr{L}}^A$ is given by

$$S_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = -\mathrm{Tr}\left[\hat{\rho}_{\mathrm{e\,out}}^{AB} \log \hat{\rho}_{\mathrm{e\,out}}^{AB}\right], \tag{17}$$

which is independent of the system B. Indeed, using the Kraus representation of the quantum channel $\hat{\mathscr{L}}^A$, we obtain the expression,

$$S_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = -\mathrm{Sp}[\mathscr{W} \log \mathscr{W}], \tag{18}$$

where $\mathscr{W}$ is a matrix whose element is given by

$$\mathscr{W}_{\mu\nu} = \mathrm{Tr}[\hat{A}_\mu \hat{\rho}^A \hat{A}_\nu^\dagger]. \tag{19}$$

Here we have used the symbol ‘Sp’ for a trace operation of matrices to distingusih it from a trace operation of operators.

Considering the unitary representation of the quantum channel and the Araki–Lieb inequality for von Neumann entropies, we find that the entropy exchange is equal to the von Neumann entropy of the environmental system at the output side of the quantum channel $\hat{\mathscr{L}}^A$,

$$S_e(\hat{\rho}^A, \hat{\mathscr{L}}^A) = S(\hat{\rho}^E_{\text{out}}) = -\text{Tr}_E\left[\hat{\rho}^E_{\text{out}} \log \hat{\rho}^E_{\text{out}}\right], \tag{20}$$

with

$$\hat{\rho}^{\text{E}}_{\text{out}} = \text{Tr}_{AB}\left|\Psi^{AB\text{E}}_{\text{out}}\right\rangle\!\left\langle\Psi^{AB\text{E}}_{\text{out}}\right|, \tag{21}$$

where $|\Psi^{AB\text{E}}_{\text{out}}\rangle$ is given by Eq. (10). The entropy exchange plays a similar role to the conditional entropy in the classical information theory. The entropy exchange $S_e(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ satisfies the quantum Fano inequality (Schumacher, 1996),

$$S_e(\hat{\rho}^A, \hat{\mathscr{L}}^A) \leq H[F_e(\hat{\rho}^A, \hat{\mathscr{L}}^A)] + [1 - F_e(\hat{\rho}^A, \hat{\mathscr{L}}^A)]\log(N^2 - 1), \tag{22}$$

where $F_e(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ is the entanglement fidelity of the quantum channel $\hat{\mathscr{L}}^A$, which is given by Eq. (12), and $H(x)$ is the binary entropic function defined by $H(x) = -x\log x - (1-x)\log(1-x)$.

The coherent information (Schumacher and Nielsen, 1996) of the quantum channel $\hat{\mathscr{L}}^A$, which is denoted as $I_C(\hat{\rho}^A, \mathscr{L}^A)$, is given by the difference between the von Neumann entropy of the output state of the system A and the entropy exchange of the quantum channel,

$$I_C(\hat{\rho}^A, \hat{\mathscr{L}}^A) = S(\hat{\rho}^A_{\text{out}}) - S_e(\hat{\rho}^A, \hat{\mathscr{L}}^A). \tag{23}$$

The coherent information $I_C(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ can take negative values and satisfies the inequality $-S(\hat{\rho}^A) \leq I_C(\hat{\rho}^A, \hat{\mathscr{L}}^A) \leq S(\hat{\rho}^A)$. Furthermore, the quantum data processing inequality is derived for the coherent information,

$$I_C(\hat{\rho}^A, \hat{\mathscr{K}}^A\hat{\mathscr{L}}^A) \leq I_C(\hat{\rho}^A, \hat{\mathscr{L}}^A), \tag{24}$$

where $\hat{\mathscr{L}}^A$ and $\hat{\mathscr{K}}^A$ are any trace-preserving completely positive maps. The loss entropy $L_C(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ and the noise entropy $N_C(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ of the quantum channel $\hat{\mathscr{L}}^A$ are defined by

$$L_C(\hat{\rho}^A, \hat{\mathscr{L}}^A) = S(\hat{\rho}^A) - I_C(\hat{\rho}^A, \hat{\mathscr{L}}^A), \tag{25}$$

$$N_C(\hat{\rho}^A, \hat{\mathscr{L}}^A) = S(\hat{\rho}^A_{\text{our}}) - I_C(\hat{\rho}^A, \hat{\mathscr{L}}^A). \tag{26}$$

It has been shown (Schumacher and Nielsen, 1996) that the quantum channel $\hat{\mathscr{L}}^A$ is completely reversible and the quantum error correction is possible if and only if $L_C(\hat{\rho}^A, \hat{\mathscr{L}}^A) = 0$. In other words, there is some quantum operation $\hat{\mathscr{R}}^A$ such that $F_e(\hat{\rho}^A, \hat{\mathscr{R}}^A\hat{\mathscr{L}}^A) = 1$ if and only if $L_C(\hat{\rho}^A, \hat{\mathscr{L}}^A) = 0$. The coherent information is considered the quantum information or the quantum entanglement transmitted through a noisy quantum channel.

4. SEPARABLE INFORMATION AND CLASSICAL CORRELATION

In this section, we consider the case that the system A in the separable state $\hat{\rho}_{\mathrm{s}}^{AB}$ closest to the entangled pure state $|\Psi^{AB}\rangle$ is sent through a noisy quantum channel $\hat{\mathcal{L}}^A$. The von Neumann entropies of the input and output states of the system A are given by Eqs. (14) and (15). In this case, the separable entropy $S_{\mathrm{s}}(\rho^A, \mathcal{L}^A)$ (Ban, 2003) is introduced by

$$S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = -\sum_k \lambda_k \mathrm{Tr}\left[\left(\hat{\mathcal{L}}^A \left|\psi_k^A\right\rangle\!\left\langle\psi_k^A\right|\right) \log\left(\hat{\mathcal{L}}^A \left|\psi_k^A\right\rangle\!\left\langle\psi_k^A\right|\right)\right], \tag{27}$$

which satisfies the quantum Fano inequality,

$$S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \leq H[\bar{F}(\hat{\rho}^A, \hat{\mathcal{L}}^A)] + [1 - \bar{F}(\hat{\rho}^A, \hat{\mathcal{L}}^A)] \log(N-1), \tag{28}$$

where $\bar{F}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ is the average fidelity of the quantum channel $\hat{\mathcal{L}}$, which is given by Eq. (13). Note that $\log(N-1)$ appears on the right-hand side of Eq. (28) while $\log(N^2-1)$ does in Eq. (22) since the correlation between the systems A and B in the separable state $\hat{\rho}_{\mathrm{s}}^{AB}$ is due to the N diagonal elements while the correlation in the entangled state $\hat{\rho}_{\mathrm{e}}^{AB}$ is caused by the N^2 diagonal and off-diagonal elements. The entropy exchange $S_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ and the separable entropy $S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ satisfies the inequality,

$$S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \leq S_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \leq S(\hat{\rho}^A) + S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A). \tag{29}$$

The separable information $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ (Ban, 2003) of the quantum channel $\hat{\mathcal{L}}^A$ is defined by the difference between the von Neumann entropy of the output state of the system A and the separable entropy of the quantum channel,

$$I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S\left(\hat{\rho}_{\mathrm{out}}^A\right) - S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A). \tag{30}$$

Unlike the coherent information, the separable information $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ is nonnegative and satisfies the inequality

$$0 \leq I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \leq \min\left[S(\hat{\rho}^A), S\left(\hat{\rho}_{\mathrm{out}}^A\right)\right]. \tag{31}$$

Furthermore, the quantum date processing inequality holds for the separable information,

$$I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{K}}^A \hat{\mathcal{L}}^A) \leq I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A), \tag{32}$$

where $\hat{\mathcal{L}}^A$ and $\hat{\mathcal{K}}^A$ are any trace-preserving completely positive maps. It is found from Eq. (29) that the separable information $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ is not less than the coherent information $I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$,

$$I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \leq I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A). \tag{33}$$

The loss entropy $L_S(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ and the noise entropy $N_S(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ in this case are given by

$$L_S(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S(\hat{\rho}^A) - I_S(\hat{\rho}^A, \hat{\mathcal{L}}^A), \tag{34}$$

$$N_S(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S(\hat{\rho}_{\text{out}}) - I_S(\hat{\rho}^A, \hat{\mathcal{L}}^A). \tag{35}$$

It is shown (Ban, 2003) that the quantum channel $\hat{\mathcal{L}}^A$ is partially reversible with respect to the orthonormal system $\{|\psi_1^A\rangle, |\psi_2^A\rangle, \ldots, |\psi_N^A\rangle\}$ if and only if $L_S(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 0$. In other words, there is a quantum operation $\hat{\mathcal{R}}^A$ such that $\bar{F}(\hat{\rho}^A, \hat{\mathcal{R}}^A\hat{\mathcal{L}}^A) = 1$ if and only if $L_S(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 0$. It is obvious from Eq. (33) that if the quantum channel is completely reversible, it is partially reversible. The separable information may be considered the measure of the transmitted classical correlation.

5. CLASSICAL CORRESPONDENCE

To obtain the classical correspondence of the entropy exchange, the coherent information, the separable entropy and the separable information, we consider a classical-like quantum, channel. A classical communication channel is completely determined by conditional probability (channel matrix) (Cover and Thomas, 1991) $P(j|k)$ that the output symbol j of the classical channel is obtained when the input symbol is k. Then a classical-like quantum channel is described by a completely positive map $\hat{\mathcal{L}}_c^A$ which corresponds to the conditional probability $P(j|k)$. In the Kraus representation, the trace-preserving completely positive map $\hat{\mathcal{L}}_c^A$ is given by

$$\hat{\mathcal{L}}_c^A \hat{\rho}^A = \sum_{j=1}^{M}\sum_{k=1}^{M} \hat{A}_{jk}\hat{\rho}^A\hat{A}_{jk}^{\dagger}, \tag{36}$$

with

$$\hat{A}_{jk} = \sqrt{P(j|k)}\big|\phi_j^B\big\rangle\big\langle\psi_k^A\big|, \tag{37}$$

where $\{|\psi_k^A\rangle | j = 1, 2, \ldots, N\}$ and $\{|\phi_j^B\rangle | k = 1, 2, \ldots, M\}$ are complete orthonormal systems of the input and output Hilbert spaces of the classical-like quantum channel. The input system which generates the symbol j with probability $P_A(k)$ is in the quantum state $\hat{\rho}^A = \sum_{k=1}^{N} P_A(k)|\psi_k^A\rangle\langle\psi_k^A|$, and the output system of the quantum channel is described by $\hat{\rho}^B = \sum_{j=1}^{M} P_B(j)|\phi_j^B\rangle\langle\phi_j^B|$, where $P_B(j) = \sum_{k=1}^{N} P(j|k)P_A(k)$. Then the input–output relation of the quantum channel is given by $\hat{\rho}^B = \hat{\mathcal{L}}_c^A\hat{\rho}^A$.

It is found for the classical-like quantum channel that the entropy exchange $S_e(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ is the joint entropy $H(A, B)$ and the separable entropy $S_s(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ is the conditional entropy $H(B|A)$, that is,

$$H(A, B) = -\sum_{j=1}^{M}\sum_{k=1}^{N} P(j|k)P_A(k)\log[P(j|k)P_A(k)], \tag{38}$$

$$H(B|A) = -\sum_{j=1}^{M}\sum_{k=1}^{N} P(j|k)P_A(k)\log P(j|k). \tag{39}$$

Then the coherent information $I_C(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ and the separable information $I_S(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ becomes $-H(A|B)$ and $H(A:B)$,

$$H(A|B) = -\sum_{j=1}^{M}\sum_{k=1}^{N} P(j|k)P_A(k)\log\left[\frac{P(j|k)P_A(k)}{P_B(j)}\right], \tag{40}$$

$$H(A:B) = -\sum_{j=1}^{M}\sum_{k=1}^{N} P(j|k)P_A(k)\log\left[\frac{P(j|k)}{P_B(j)}\right], \tag{41}$$

where $H(A:B)$ is the Shannon mutual information. Summarizing the result, we obtain the relations for the classical-like quantum channel,

$$S_e(\hat{\rho}^A, \hat{\mathcal{L}}^A) = H(A, B), \tag{42}$$

$$S_s(\hat{\rho}^A, \hat{\mathcal{L}}^A) = H(B|A), \tag{43}$$

$$S_C(\hat{\rho}^A, \hat{\mathcal{L}}^A) = -H(A|B), \tag{44}$$

$$S_S(\hat{\rho}^A, \hat{\mathcal{L}}^A) = H(A:B). \tag{45}$$

6. DEGRADATION OF CORRELATIONS

The quantum state $\hat{\rho}_s^{AB}$ is the separable state closest to the entangled pure state $\hat{\rho}_e$ when the distance between the quantum states is measured by the quantum relative entropy. The distance $E(\hat{\rho}^A, \hat{\mathcal{I}}^A)$ between the separable state $\hat{\rho}_s^{AB}$ and the entangled state $\hat{\rho}_e^{AB}$ is given by

$$E(\hat{\rho}^A, \hat{\mathcal{I}}^A) = S\big(\hat{\rho}_e^{AB}|\hat{\rho}_s^{AB}\big) = S(\hat{\rho}^A), \tag{46}$$

which is equal to the entropy of entanglement (Bennett, *et al.*, 1996) of the pure entangled state $|\Psi^{AB}\rangle$. Thus the distance $E(\hat{\rho}^A, \hat{\mathcal{I}}^A)$ is the entanglement measure of the quantum state $\hat{\rho}_e^{AB}$. When the system A in the quantum state $\hat{\rho}_e^{AB}$ or $\hat{\rho}_s^{AB}$ is transmitted through the quantum channel $\hat{\mathcal{L}}^A$, the correlation between the system A and the system B is inevitably degraded under the influence of the environmental system. The distance between the transmitted quantum states becomes

$$\begin{aligned} E(\hat{\rho}^A, \hat{\mathcal{L}}^A) &= S\big(\hat{\rho}_{e\,out}^{AB}|\hat{\rho}_{s\,out}^{AB}\big) \\ &= S(\hat{\rho}^A) + I_C(\hat{\rho}^A, \hat{\mathcal{L}}^A) - I_S(\hat{\rho}^A, \hat{\mathcal{L}}^A). \end{aligned} \tag{47}$$

Then the decrease of the distance caused by the quantum channel $\hat{\mathcal{L}}^A$ is given by the difference between the separable information and the coherent information or the difference between the separable entropy and the entropy exchange,

$$\begin{aligned}\Delta E(\hat{\rho}^A, \hat{\mathcal{L}}^A) &= E(\hat{\rho}^A, \hat{\mathcal{I}}^A) - E(\hat{\rho}^A, \hat{\mathcal{L}}^A) \\ &= I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) - I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \\ &= S_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) - S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A). \end{aligned} \tag{48}$$

Since the quantum relative entropy $S(\hat{\rho}_1|\hat{\rho}_2)$ satisfies the inequality for any trace-preserving completely positive map $\hat{\mathcal{L}}$,

$$S(\hat{\rho}_1|\hat{\rho}_2) \geq S(\hat{\mathcal{L}}\hat{\rho}_1|\hat{\mathcal{L}}\hat{\rho}_2), \tag{49}$$

the quantity $\Delta E(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ is nonnegative. This result means that the inequalities $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \geq I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ and $S_{\mathrm{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) \leq S_{\mathrm{e}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ hold.

We now consider simple quantum channels. A unitary channel is given by $\hat{\mathcal{L}}^A \hat{X}^A = \hat{U}^A \hat{X}^A \hat{U}^{A\dagger}$, where $\hat{U}^A$ is a unitary operator. In this case, we obtain

$$I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S(\hat{\rho}^A), \tag{50}$$

$$L_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = L_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 0 \tag{51}$$

$$N_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = N_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 0. \tag{52}$$

Thus the unitary channel is lossless and noiseless and thus it does not decrease the distance between the quantum states $\hat{\rho}_{\mathrm{e}}^{AB}$ and $\hat{\rho}_{\mathrm{e}}^{AB}$, namely, $\Delta E(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 0$ Next we consider a completely randomizing channel defined by $\hat{\mathcal{L}}^A \hat{\rho}^A = \hat{\xi}^A$, where $\hat{\xi}^A$ is some quantum state such as $\hat{\xi}^A = (1/N)\hat{1}^A$. The output state of the completely randomizing channel is independent of the input state. Then we obtain

$$I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = -S(\hat{\rho}^A), \tag{53}$$

$$L_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 2S(\hat{\rho}^A), \tag{54}$$

$$N_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 2S(\hat{\xi}^A), \tag{55}$$

and

$$I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = 0, \tag{56}$$

$$L_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S(\hat{\rho}^A), \tag{57}$$

$$N_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S(\hat{\xi}^A). \tag{58}$$

where $S(\hat{\xi}^A)$ is the von Neumann entropy of the quantum state $\hat{\xi}^A$. In this case, the decrease of the distance between the quantum states $\hat{\rho}_{\mathrm{e}}^{AB}$ and $\hat{\rho}_{\mathrm{e}}^{AB}$ is given by $\Delta E(\hat{\rho}^A, \hat{\mathcal{L}}^A) = S(\hat{\rho}^A)$.

7. VON NEUMANN MUTUAL INFORMATION OF QUANTUM CHANNEL

First we briefly review the Shannon mutual information for a classical channel. An input system of a classical channel is described by a stochastic variable A which takes a value $a \in \Omega_A$ with probability $P_A(a)$. An output system is described by another stochastic variable B which takes a value $b \in \Omega_B$ with probability $P_B(b)$. The classical channel creates some classical correlation between the input and output systems. Then the compound system which consists of the input and output system is described by the stochastic variables A and B with the joint probability $P_{AB}(a, b)$ which must satisfy $P_B(b) = \sum_{a\in\Omega_A} P_{AB}(a, b)$ and $P_A(a) = \sum_{b\in\Omega_B} P_{AB}(a, b)$. The degree of the correlation between the input and output system depends on how much information is transmitted from the input system to the output system by the classical channel. If there is no information transmission, there is no correlation and thus the equality $P_{AB}(a, b) = P_A(a)P_B(b)$ holds. Hence the Shannon mutual information $I_C(A : B)$ between the input and output systems may be measured by means of the classical relative entropy of the two probabilities $P_{AB}(a, b)$ and $P_A(a)P_B(b)$, that is,

$$
\begin{aligned}
I_C(A : B) &= \sum_{a\in\Omega_A} \sum_{b\in\Omega_B} P_{AB}(a, b) \log \left[\frac{P_{AB}(a, b)}{P_A(a)P_B(b)} \right] \\
&= H(A) + H(B) - H(A, B),
\end{aligned} \tag{59}
$$

where $H(A)$, $H(B)$, and $H(A, B)$ are the Shannon entropies of $P_A(a)$, $P_B(b)$, and $P_{AB}(a, b)$, for example, $H(A) = \sum_{a\in\Omega_A} P_A(a) \log P_A(a)$.

For a quantum channel, we suppose that an input system of the quantum channel is described by a quantum state $\hat{\rho}^A$ and an output system is described by another quantum state $\hat{\rho}^B$. A quantum channel which can transmit quantum and classical information from the input system from the output system makes some correlation between them. Then the compound system consisting of the input and output systems is described by a quantum state $\hat{\rho}^{AB}$ which satisfies the relations $\hat{\rho}^A = \mathrm{Tr}_B \hat{\rho}^{AB}$ and $\hat{\rho}^B = \mathrm{Tr}_A \hat{\rho}^{AB}$. If there is no information transmission, it is obvious that the equality $\hat{\rho}^{AB} = \hat{\rho}^A \otimes \hat{\rho}^B$ holds. Thus the von Neumann mutual information $I_Q(A : B)$ between the input and output systems is given by the quantum relative entropy of the quantum states $\hat{\rho}^{AB}$ and $\hat{\rho}^A \otimes \hat{\rho}^B$, that is,

$$
\begin{aligned}
I_Q(A : B) &= \mathrm{Tr}_{AB}(\hat{\rho}^{AB}[\log \hat{\rho}^{AB} - \log(\hat{\rho}^A \otimes \hat{\rho}^B)]) \\
&= S(\hat{\rho}^A) + S(\hat{\rho}^B) - S(\hat{\rho}^{AB}).
\end{aligned} \tag{60}
$$

It is important to note that the compound quantum state $\hat{\rho}^{AB}$ cannot be determined uniquely. It may depend on what we consider as information transmitted through the quantum channel.

We first consider the case that the compound state is given by

$$\begin{aligned}\hat{\rho}^{AB}_{\text{e out}} &= (\hat{\mathcal{L}}^A \otimes \hat{\mathcal{I}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}| \\ &= \sum_{j=1}^{N}\sum_{k=1}^{N}\sqrt{\lambda_j\lambda_k}\hat{\mathcal{L}}^A\big|\psi_j^A\big\rangle\big\langle\psi_k^A\big| \otimes \big|\psi_j^B\big\rangle\big\langle\psi_k^B\big|,\end{aligned} \tag{61}$$

where $\psi^A = \sum_k \lambda_k|\psi_k^A\rangle\langle\psi_k^A|$ is the spectral decomposition. In this case, the von Naumann mutual information $I_Q(A:B)$ between the input and output systems is given by the coherent information of the quantum channel and the von Neumann entropy of the input system, that is,

$$I_{\mathrm{Q}}(A:B) = S(\hat{\rho}^A) + I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A). \tag{62}$$

On the other hand, if the classical correlation is considered the information transmitted through the quantum channel $\hat{\mathcal{L}}^A$, the compound state is given by the separable state,

$$\hat{\rho}^{AB}_{\text{s out}} = \sum_{k=1}^{N}\lambda_k\hat{\mathcal{L}}^A\big|\psi_k^A\big\rangle\big\langle\psi_k^A\big| \otimes \big|\psi_k^B\big\rangle\big\langle\psi_k^B\big|, \tag{63}$$

which is equivalent to that introduced in (Ahlswede, and Löber, 2001; Ohya and Petz, 1993). Then, the von Neumann mutual information $I_Q(A:B)$ becomes the separable information,

$$I_Q(A:B) = I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A). \tag{64}$$

When we transmit classical information by sending the quantum state $|\psi_k^A\rangle\langle\psi_k^A|$ with probability λ_k through the quantum channel $\hat{\mathcal{L}}^A$, the classical capacity of a quantum channel (Holevo, 1998; Schumacher and Westmore-land, 1997) (the Holevo capacity) C_{H} is given by

$$\begin{aligned}C_{\mathrm{H}} &= S\left[\sum_{k=1}^{N}\lambda_k\hat{\mathcal{L}}^A\big|\psi_k^A\big\rangle\big\langle\psi_k^A\big|\right] - \sum_{k=1}^{N}\lambda_k S\big[\hat{\mathcal{L}}^A\big|\psi_k^A\big\rangle\big\langle\psi_k^A\big|\big] \\ &= I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A),\end{aligned} \tag{65}$$

which is optimized as $\max_{\hat{\rho}^A} I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$. On the other hand, the entanglement-assisted classical capacity of a quantum channel is given by $\max_{\hat{\rho}^A}[S(\hat{\rho}^A) + I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)]$ (Bennett *et al.*, 2002). Both of the coherent information and the separable information are related to the classical capacity of a quantum channel.

8. LINEAR DISSIPATIVE CHANNEL OF QUBITS

In this section, we calculate the coherent information and separable information of the linear dissipative channel of qubits. When we denote as t a time during which a qubit is transmitted through the quantum channel, the linear dissipative channel (Gardiner, 1991) of qubits is given by

$$\hat{\mathscr{L}}\hat{X} = \exp[t\hat{\Lambda}]\hat{X}, \tag{66}$$

with

$$\hat{\Lambda}\hat{X} = \frac{1}{2}\lambda_{10}([\hat{\sigma}_-\hat{X}, \hat{\sigma}_+] + [\hat{\sigma}_-, \hat{X}\hat{\sigma}_+]) + \frac{1}{2}\lambda_{01}([\hat{\sigma}_+\hat{X}, \hat{\sigma}_-] + [\hat{\sigma}_+, \hat{X}\hat{\sigma}_-]), \tag{67}$$

where $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$ and $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the Pauli matrices, and the parameter γ_{jk} is the damping constant of the quantum channel. For the sake of simplicity, we assume that $\gamma_{10} = \gamma_{01} \equiv \gamma$. In the Kraus representation, the linear dissipative channel is expressed as

$$\hat{\mathscr{L}}\hat{X} = \sum_{\mu=1}^{4} \hat{A}_\mu \hat{X} \hat{A}_\mu^\dagger, \tag{68}$$

with

$$\hat{A}_1 = \frac{1}{2}(1 + e^{-\gamma t})\hat{1}, \tag{69}$$

$$\hat{A}_2 = \frac{1}{2}\sqrt{1 - e^{-2\gamma t}}\hat{\sigma}_x, \tag{70}$$

$$\hat{A}_3 = \frac{1}{2}\sqrt{1 - e^{-2\gamma t}}\hat{\sigma}_y, \tag{71}$$

$$\hat{A}_4 = \frac{1}{2}(1 + e^{-\gamma t})\hat{\sigma}_z. \tag{72}$$

For the eigenstates $|0\rangle$ and $|1\rangle$ of the Pauli matrix $\hat{\sigma}_z$ with $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$, the linear dissipative channel $\hat{\mathscr{L}}$ yields the relations,

$$\hat{\mathscr{L}}|0\rangle\langle 0| = \frac{1}{2}(1 + e^{-2\gamma t})|0\rangle\langle 0| + \frac{1}{2}(1 - e^{-2\gamma t})|1\rangle\langle 1|, \tag{73}$$

$$\hat{\mathscr{L}}|1\rangle\langle 1| = \frac{1}{2}(1 - e^{-2\gamma t})|0\rangle\langle 0| + \frac{1}{2}(1 + e^{-2\gamma t})|1\rangle\langle 1|, \tag{74}$$

$$\hat{\mathscr{L}}|0\rangle\langle 1| = e^{-\gamma t}|0\rangle\langle 1|, \tag{75}$$

$$\hat{\mathscr{L}}|1\rangle\langle 0| = e^{-\gamma t}|1\rangle\langle 0|. \tag{76}$$

We assume that the input system A of the quantum channel takes an orthogonal quantum state $|\psi_0\rangle$ or $|\psi_1\rangle$ with equal probabilites, where

$$|\psi_0\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle, \tag{77}$$

$$|\psi_1\rangle = -\sin\theta|0\rangle + e^{i\phi}\cos\theta|1\rangle. \tag{78}$$

Then the quantum state of the input system A is completely random,

$$\hat{\rho}^A = \frac{1}{2}|\psi_0\rangle\langle\psi_0| + \frac{1}{2}|\psi_1\rangle\langle\psi_1| = \frac{1}{2}\hat{1}. \tag{79}$$

Since $\hat{\rho}^A_{\text{out}} = \hat{\mathcal{L}}\hat{\rho}^A = \hat{\rho}^A$, the von Neumann entropies of the input and output states are equal and given by

$$S\big(\hat{\rho}^A_{\text{out}}\big) = S(\hat{\rho}^A) = \log 2. \tag{80}$$

The entangled state $|\Psi^{AB}\rangle$ and the separable state $\hat{\rho}^{AB}_{\text{s}}$ closest to $|\Psi^{AB}\rangle$ are

$$|\Psi^{AB}\rangle = \frac{1}{\sqrt{2}}(|\psi_0\rangle\otimes|\phi_0\rangle + |\psi_1\rangle\otimes|\phi_1\rangle), \tag{81}$$

$$\hat{\rho}^{AB}_{\text{s}} = \frac{1}{2}|\psi_0\rangle\langle\psi_0|\otimes|\phi_0\rangle\langle\phi_0| + \frac{1}{2}|\psi_1\rangle\langle\psi_1|\otimes|\phi_1\rangle\langle\phi_1|, \tag{82}$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthonormal states of the system B.

To obtain the coherent information $I_{\text{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$, we need the quantum state $\hat{\rho}^{AB}_{\text{e out}} = (\mathcal{L}^A \otimes \mathcal{I}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|$. In the matrix notation with the base $|j\rangle\otimes|\phi_k\rangle$ $(j, k = 0, 1)$, the quantum state $\hat{\rho}^{AB}_{\text{e out}}$ is given by

$$\hat{\rho}^{AB}_{\text{e out}} = \frac{1}{2}\begin{pmatrix} A_+ & B_- & D_+ & C_+ \\ B_- & A_- & C_- & D_- \\ D_+^* & C_-^* & A_- & B_+ \\ C_+^* & D_-^* & B_+ & A_+ \end{pmatrix}, \tag{83}$$

with

$$A_\pm = \frac{1}{2}(1 \pm e^{-2\gamma t}\cos 2\theta), \tag{84}$$

$$B_\pm = \pm\frac{1}{2}e^{-2\gamma t}\sin 2\theta, \tag{85}$$

$$C_\pm = \pm\frac{1}{2}e^{-\gamma t}(1 \pm \cos 2\theta)e^{i\phi}, \tag{86}$$

$$D_\pm = \pm\frac{1}{2}e^{-i\phi}e^{-\gamma t}\sin 2\theta. \tag{87}$$

The eigenvalues of the matrix (83) are $\frac{1}{4}(1 - e^{-2\gamma t})$, $\frac{1}{4}(1 - e^{-2\gamma t})$, $\frac{1}{4}(1 + e^{-\gamma t})^2$, and $\frac{1}{4}(1 - e^{-\gamma t})^2$. Hence the coherent information $I_{\text{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ of the linear dissipative channel of qubits becomes

$$I_{\text{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = -\log 2 + (1 + e^{-\gamma t})\log(1 + e^{-\gamma t}) + (1 - e^{-\gamma t})\log(1 - e^{-\gamma t}). \tag{88}$$

Next we obtain the separable information $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ of the linear dissipative channel. We find from Eqs. (73)–(78) that

$$\hat{\mathscr{L}}|\psi_0\rangle\langle\psi_0| = A_+|0\rangle\langle 0| + A_-|1\rangle\langle 1| + B_+(e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|), \quad (89)$$

$$\hat{\mathscr{L}}|\psi_1\rangle\langle\psi_1| = A_-|0\rangle\langle 0| + A_+|1\rangle\langle 1| + B_-(e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|). \quad (90)$$

The operators $\hat{\mathscr{L}}|\psi_0\rangle\langle\psi_0|$ and $\hat{\mathscr{L}}|\psi_1\rangle\langle\psi_1|$ have the same eigenvalues,

$$\lambda_\pm = \frac{1}{2}[1 \pm e^{-\gamma t} f(\cos^2 2\theta)], \quad (91)$$

with

$$f(x) = \sqrt{1 - (1 - e^{-2\gamma t})x}. \quad (92)$$

Therefore the separable information $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ is given by

$$I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \frac{1}{2}[1 + e^{-\gamma t} f(\cos^2 2\theta)] \log[1 + e^{-\gamma t} f(\cos^2 2\theta)] + \frac{1}{2}[1 - e^{-\gamma t} f(\cos^2 2\theta)] \log[1 - e^{-\gamma t} f(\cos^2 2\theta)]. \quad (93)$$

Note that the coherent information $I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ is independent of the parameters θ and ϕ while the separable information $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ depends on the parameter θ. In the case of the quantum depolarizing channel $\hat{\mathscr{L}}\hat{X} = \sum_{\mu=1}^{4} \hat{A}_\mu \hat{X} \hat{A}_\mu^\dagger$ with $\hat{A}_1 = \sqrt{1-p}\hat{1}$, $\hat{A}_2 = \sqrt{p/3}\hat{\sigma}_x$, $\hat{A}_3 = \sqrt{p/3}\hat{\sigma}_y$, and $\hat{A}_4 = \sqrt{p/3}\hat{\sigma}_z (0 \le p \le 1)$, the separable informaton $I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ becomes independent of the parameter θ because of the symmetric property of the quantum depolarizing channel.

It is easy to see that the maximum value of the separable information with respect to the parameter θ are

$$I_{\mathrm{S}}^{\max}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \max_\theta I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \frac{1}{2}(1 + e^{-\gamma t}) \log(1 + e^{-\gamma t}) + \frac{1}{2}(1 - e^{-\gamma t}) \log(1 - e^{-\gamma t}), \quad (94)$$

which is attained when the input system takes the quantum states

$$|\psi_0\rangle = \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}, \qquad |\psi_1\rangle = \frac{|0\rangle - e^{i\phi}|1\rangle}{\sqrt{2}}. \quad (95)$$

On the other hand, when the input system takes the quantum states $|0\rangle$ and $|1\rangle$, the minimum value of of the separable information is given by

$$I_{\mathrm{S}}^{\max}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \max_\theta I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \frac{1}{2}(1 + e^{-\gamma t}) \log(1 + e^{-\gamma t}) + \frac{1}{2}(1 - e^{-\gamma t}) \log(1 - e^{-\gamma t}), \quad (96)$$

This result means that when the input system takes the superposition state, the classical correlation between the input and output system becomes stronger for the linear dissipative channel of qubits.

9. QUANTUM ERASURE CHANNEL

The quantum erasure channel transform a input quantum state into another quantum state with probability p which is defined on an orthogonal subspace of the support space of the input quantum state and makes the input state unchanged with probability $1 - p$. When the orthogonal subspace is an one-dimensional space spanned by $|\phi\rangle$, the completely positive map which describes the quantum erasure channel is given by

$$\hat{\mathcal{L}}^A \hat{\rho}^A = (1 - p)\hat{\rho}^A + p|\phi\rangle\langle\phi|, \tag{97}$$

Where $0 \leq p \leq 1$ and $\hat{\rho}^A|\phi\rangle\langle\phi| = |\phi\rangle\langle\phi|\hat{\rho}^A = 0$. Then the von Neumann entropy of the output quantum state becomes

$$S\left(\hat{\rho}^A_{\text{out}}\right) = H(p) + (1 - p)S(\hat{\rho}^A), \tag{98}$$

with $H(p) = -(1 - p)\log(1 - p) - p\log p$. Furthermore, the entropy exchange $S_{\text{e}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ and the separable entropy $S_{\text{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ are calculated to be

$$S_{\text{e}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = H(p) + pS(\hat{\rho}^A), \tag{99}$$

$$S_{\text{s}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = H(p). \tag{100}$$

Hence the coherent information $I_{\text{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ and the separable information $I_{\text{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A)$ of the quantum erasure channel become

$$I_{\text{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = (1 - 2p)S(\hat{\rho}^A), \tag{101}$$

$$I_{\text{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = (1 - p)S(\hat{\rho}^A). \tag{102}$$

If the erasure probability p is larger than 1/2, the coherent information takes a negative value. When the Hilbert space of the system is an n-dimensional space, we obtain

$$\max_{\hat{\rho}^A} I_{\text{C}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = \begin{cases} (1 - 2p)\log n & \left(0 \leq p \leq \dfrac{1}{2}\right) \\ 0 & \left(\dfrac{1}{2} \leq p \leq 1\right) \end{cases}, \tag{103}$$

$$\max_{\hat{\rho}^A} I_{\text{S}}(\hat{\rho}^A, \hat{\mathcal{L}}^A) = (1 - p)\log n. \tag{104}$$

In this case, the decrease of the distance $\Delta E(\hat{\rho}^A, \hat{\mathscr{L}}^A)$ given by Eq. (48) becomes

$$\Delta E(\hat{\rho}^A, \hat{\mathscr{L}}^A) = \begin{cases} p \log n & \left(0 \le p \le \frac{1}{2}\right) \\ (1-p)\log n & \left(\frac{1}{2} \le p \le 1\right) \end{cases}, \tag{105}$$

which takes the maximum value at $p = 1/2$.

Next we consider the phase-erasure quantum channel of qubits that is described by a completely positive map,

$$\hat{\mathscr{L}}^A \hat{\rho}^A = (1-p)\hat{\rho}^A + \frac{1}{2} p(\hat{\rho}^A + \hat{\sigma}_z \hat{\rho}^A \hat{\sigma}_z), \tag{106}$$

with $0 \le p \le 1$. An arbitrary quantum state $\hat{\rho}^A$ of the input system is expressed as

$$\hat{\rho}^A = \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & re^{i\phi}\sin\theta \\ re^{-i\phi}\sin\theta & 1 - r\cos\theta \end{pmatrix}, \tag{107}$$

where $0 \le r \le 1$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. Then, after some calculation, we obtain the coherent information and the separable information,

$$I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = (1-p)H\left(\frac{1+r}{2}\right), \tag{108}$$

$$I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) + p\left[H\left(\frac{1+r\cos\theta}{2}\right) - H\left(\frac{1+\cos\theta}{2}\right)\right], \tag{109}$$

the maximum values of which are given by

$$\max_{0\le r\le 1, 0\le\theta\le\pi} I_{\mathrm{C}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = 1 - p, \tag{110}$$

$$\max_{0\le r\le 1, 0\le\theta\le\pi} I_{\mathrm{S}}(\hat{\rho}^A, \hat{\mathscr{L}}^A) = 1. \tag{111}$$

The maximum value of the separable information becomes independent of p by setting the orthonormal system such that the off-diagonal elements destroyed by the quantum channel does not appear.

10. CONCLUDING REMARKS

In this paper, we have considered the quantum and classical correlation in quantum channels by means of the information-theoretical quantities, the coherent information and the separable information. The degradation of the correlation caused by noisy quantum channels has also been studied. As the examples, we have investigated the linear dissipative channel of qubits and the quantum erasure channel.

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